Anti-Kählerian geometry on Lie groups

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Preliminaries

Definition (Almost anti-Hermitian Manifold)

An *almost anti-Hermitian manifold* is a triple (M, g, J), where M is a differentiable manifold of real dimension $2n$, J is an almost complex structure on M and g is an *anti-Hermitian* metric on (M, J); that is

$$g(JX, JY) = -g(X, Y), \forall X, Y \in \mathfrak{X}(M), \tag{1}$$

or equivalently, J is symmetric with respect to g.

If additionally, J is integrable, then the triple (M, g, J) is called *anti-Hermitian manifold* or *complex Norden manifold*.

Signature

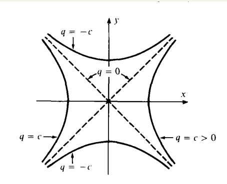
If (M, g, J) is an almost anti-Hermitian manifold, it is straightforward to check that the signature of g is (n, n); i.e. g is a neutral metric.

The linear (algebra) model of an almost anti-Hermitian manifold is given by

1. $\mathbb{R}^{n,n} := (\mathbb{R}^{2n}, \varphi = (x_1, y_1, \ldots, x_n, y_n), \langle \cdot, \cdot \rangle)$,
2. Complex structure: $Jx_i = y_i$ y $Jy_i = -x_i$,
3. Inner product: $\langle x_i, x_j \rangle = \delta_{i,j}$, $\langle y_i, y_j \rangle = -\delta_{i,j}$, $\langle x_i, y_j \rangle = 0$.

Equivalently, $(\mathbb{C}^n, \widehat{\varphi} = (z_1, \ldots, z_n))$ with the $\mathbb{C}$-symmetric inner product $\langle\langle \cdot, \cdot \rangle\rangle$

$$\langle\langle z, z \rangle\rangle = z_1^2 + \ldots + z_n^2.$$



Definition (Anti-Kähler manifold)

An *Anti-Kähler manifold* is an almost anti-Hermitian manifold (M, g, J) such that J is parallel with respect to the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

Let (M, g, J) be an almost anti-Hermitian manifold. From now on, let us denote by ∇ the Levi-Civita connection of (M, g)

Integrability

Note that an anti-Kähler manifold (M, g, J) satisfies that J is integrable:

$$\begin{aligned} N(X, Y) &:= [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \\ &= (\nabla_{JX} J)Y - J(\nabla_X J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X, \end{aligned} \tag{2}$$

for all X, Y in $\mathfrak{X}(M)$.

Lemma

Let (M, g, J) be an almost anti-Hermitian manifold. Then $(\nabla_X J)$ is a symmetric operator with respect to the metric g; i.e.

$$g((\nabla_X J)Y, Z) = g(Y, (\nabla_X J)Z),\ \forall X, Y, Z \in \mathfrak{X}(M). \tag{3}$$

Lemma

Let (M, g, J) be an almost anti-Hermitian manifold. Then, (M, g, J) is an anti-Kähler manifold if and only if

$$(\nabla_{JX} J)Y = \varepsilon J(\nabla_X J)Y,\ \forall X, Y \in \mathfrak{X}(M) \tag{4}$$

where ε is a real constant.

Definition (Twin metric)

Let (M, g, J) be an almost anti-Hermitian manifold. The tensor defined by the formula $\widetilde{g}(X, Y) := g(JX, Y)$, $\forall X, Y \in \mathfrak{X}(M)$ is symmetric because of equation (1), we have even more, $(M, \widetilde{g}, J)$ is an almost anti-Hermitian manifold. The metric $\widetilde{g}$ is called *associated metric*, *twin metric* or *dual metric*.

The Levi-Civita connection of g and $\widetilde{g}$

Let (M, g, J) be an anti-Kähler manifold and $\widetilde{g}$ its twin metric. The Levi-Civita connection of the twin metric coincides with the Levi-Civita connection of g. In particular $(M, \widetilde{g}, J)$ is also an anti-Kähler manifold.

Pureness

By using before fact, it follows that the Riemannian curvature tensor of an anti-Kähler manifold (M, g, J) is *pure*; i.e. for smooth vector fields X, Y, Z, W

$$\begin{aligned} R(JX, Y, Z, W) &= R(X, JY, Z, W) \\ &= R(X, Y, JZ, W) = R(X, Y, Z, JW). \end{aligned}$$

Left invariant geometric structures on Lie groups

We now proceed to consider Lie groups endowed with left invariant geometry structures. Let G be a Lie group and let us denote by $\mathfrak{g}$ its Lie algebra, which is the finite dimensional real vector space consisting of all smooth vector fields invariant under left translations L_p, $p \in G$. If g is a left invariant pseudo-Riemannian metric on G; i.e. the left translations are isometries of (G, g), then g is completely determined by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by g:

$$\langle X, Y \rangle \quad = \quad g(X, Y), \forall X, Y \in \mathfrak{g}.$$

An almost complex structure J on a Lie group G is said to be *left invariant* if $(\mathrm{d}\, L_p) \circ J = J \circ (\mathrm{d}\, L_p)$ for all $p \in G$; equivalently for all $X \in \mathfrak{g}$, $J \circ X \in \mathfrak{g}$.

Proposition

Let G be a $2n$-dimensional Lie group.

1. *There exist a left invariant metric g on G and a left invariant almost complex structure J on G such that (G, g, J) is an almost anti-Hermitian manifold.*
2. *If J is an almost left invariant complex structure on G, then there exists a left invariant metric g such that (G, g, J) is an almost anti-Hermitian manifold.*
3. *If g is a left invariant metric of signature (n, n) on G, then there exists an almost left invariant complex structure J on G such that (G, g, J) is an almost anti-Hermitian manifold.*

Definition (Abelian complex structure)

A left invariant almost complex structure J on a Lie Group G is called *abelian* when it satisfies

$$[JX, JY] = [X, Y], \forall X, Y \in \mathfrak{g}. \tag{5}$$

Note that an abelian complex structure J on a Lie group G is in fact integrable, hence (G, J) is a complex manifold, but (G, J) is not a complex Lie group (unless G is a commutative Lie group when it is connected).

Definition (Bi-invariant complex structure)

A left invariant almost complex structure J on a Lie group is called *bi-invariant* if it satisfies

$$[JX, Y] = J[X, Y](= [X, JY]), \forall X, Y \in \mathfrak{g}. \tag{6}$$

Note that a bi-invariant complex structure J on a Lie group G is in fact integrable, and even more, (G, J) is a complex Lie group.

Left invariant anti-Kähler structures on Lie groups

Proposition

Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G. If any of the following conditions are satisfied:

$$\nabla_{JX}Y = -J\nabla_X Y, \ \forall X, Y \in \mathfrak{g}, \tag{7}$$

$$\nabla_{JX}Y = J\nabla_X Y, \ \forall X, Y \in \mathfrak{g}, \tag{8}$$

then (G, g, J) is an anti-Kähler manifold, and even more so, J is an abelian complex structure in the case of the condition (7) and J is a bi-invariant complex structure in the case of the condition (8).

Proposition

Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is abelian complex structure. Then (G, g, J) satisfies the condition (7); i.e.

$$\nabla_{JX}Y = -J\nabla_X Y, \forall X, Y \in \mathfrak{g}$$

Proposition

Let (g, J) a left invariant anti-Kähler structure on a Lie group G such that J is bi-invariant complex structure. Then (G, g, J) satisfies the condition (8); i.e.

$$\nabla_{JX}Y = J\nabla_X Y, \ \ \forall X, Y \in \mathfrak{g}$$

Anti-Kähler geometry on complex Lie groups

Proposition

Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G with J being a bi-invariant complex structure on G. Then (G, g, J) is an anti-Kähler manifold.

Wild classification

Combining before propositions with well-known results of representation of algebras and *wild problems* (also known as *hopeless problems*), we have that the classification of anti-Kähler manifolds could be a wild problem.

Proposition

Let G be a Lie group admitting a left invariant anti-Kähler-Einstein structure (g, J) with non-vanishing cosmological constant and g being a bi-invariant metric. Then, G is a semisimple Lie group and J is a bi-invariant complex structure on G.

Anti-Kähler geometry and abelian complex structures

Proposition

Let G be a Lie group admitting a left invariant anti-Kähler structure (g, J) with J being an abelian complex structure on G. Then the Levi-Civita connection of (G, g) is completely determined just by the complex structure J and the Lie algebra $\mathfrak{g}$:

$$\nabla_X Y = \frac{1}{2}\left([X, Y] - J[X, JY]\right), \forall X, Y \in \mathfrak{g} \tag{9}$$

Proposition (Obstruction)

If G is a Lie group admitting a left invariant anti-Kähler structure (g, J) with J being an abelian complex structure, then $\mathfrak{g}$ is a unimodular Lie algebra; i.e. forall X in $\mathfrak{g}$

$$\mathrm{Tr}(\mathrm{ad}_X) = 0.$$

Corollary

If G is a Lie group admitting a left invariant anti-Kähler structure (g, J) with J being an abelian complex structure, then

$$B(JX, JY) = -B(X, Y), \forall X, Y \in \mathfrak{g},$$

where B is the Killing form of $\mathfrak{g}$.

Curvature

Proposition

If G is a Lie group admitting a left invariant anti-Kähler structure (g, J) with J being an abelian complex structure, then for all X, Y, Z in $\mathfrak{g}$

$$\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z.$$

Theorem

*Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure. Then (G, g) is a **flat pseudo-Riemannian manifold**.*

Corollary (Obstruction)

Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure, then for all X, Y, Z in $\mathfrak{g}$

$$[J[X, Y], Z] \quad = \quad J[[X, Y], Z].$$

Some 3-forms associated with left invariant anti-Kähler structures on Lie groups

Let G be a Lie group admitting a left invariant anti-Kähler structure (g, J). We begin here by defining a family of bilinear maps on $\mathfrak{g}$ in the following way: let $\{a_1, \ldots, a_4\}$ be real constants and $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the bilinear map on $\mathfrak{g}$ given by

$$B(X, Y) \quad = \quad a_1 \nabla_X Y + a_2 \nabla_{JX} Y + a_3 J \nabla_X Y + a_4 J \nabla_{JX} Y.$$

and let β be the covariant 3-tensor on $\mathfrak{g}$ defined by

$$\beta(X, Y, Z) \quad = \quad \langle B(X, Y), Z \rangle, \ \forall X, Y, Z \in \mathfrak{g}.$$

The skew-symmetric part of β is a multiple of

$$\theta(X, Y, Z) \quad = \quad \langle B(X, Y), Z \rangle + \langle B(Y, Z), X \rangle + \langle B(Z, X), Y \rangle \tag{10}$$

We want to highlight an important member in this family of skew-symmetric tensors which is defined from the particular bilinear map

$$B(X, Y) = \nabla_{JX} Y + J\nabla_X Y. \tag{11}$$

In this case, we have that β is a *pure* tensor on $\mathfrak{g}$, and so θ is a pure skew-symmetric 3-tensor on $\mathfrak{g}$.

Now, we consider on $\mathfrak{g}$ the complex vector space structure induced by J; $(a + \sqrt{-1}\, b) \cdot X := aX + JbX$ and $\widehat{\theta}(X, Y, Z) = \theta(X, Y, Z) - \sqrt{-1}\, \theta(JX, Y, Z)$. In this way, we have on $(\mathfrak{g}, \mathbb{C})$ a complex skew-symmetric 3-tensor and θ is its real part.

Furthemore, θ has the following very nice expression: for all X, Y, Z in $\mathfrak{g}$

$$\theta(X, Y, Z) = \langle [JX, Y], Z\rangle + \langle [JY, Z], X\rangle + \langle [JZ, X], Y\rangle. \tag{12}$$

Theorem

Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G and let θ be its 3-tensor as is defined in (10) from (11). Then, (G, g, J) is an anti-Kähler manifold if and only if θ is skew-symmetric and pure on $\mathfrak{g}$; equivalently,

$$\theta(X, Y, Z) = -\theta(X, Z, Y) \tag{13}$$

and

$$\theta(JX, Y, Z) = \theta(X, JY, Z) \tag{14}$$

for all X, Y, Z in $\mathfrak{g}$.

Corollary

Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G such that its 3-tensor θ vanishes identically on $\mathfrak{g}$. Then (G, g, J) is an anti-Kähler manifold.

Corollary

Let (g, J) be a left invariant almost anti-Hermitian structure on 4-dimensional Lie group G. Then, (G, g, J) is an anti-Kähler manifold if and only if its 3-tensor θ as is defined in (10) vanishes identically on $\mathfrak{g}$.

Left invariant anti-Kähler structures on four dimensional Lie groups

Let G be a real Lie group of dimension four which admits a left invariant anti-Kähler structure (g, J). As before, let us denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{g}$ induced by the left invariant metric g on G.

There exists an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G of the form $\mathcal{B} = \{X, JX, Y, JY\}$ with X and Y are spacelike, and so, we have

$$\left[g\right]_{\mathcal{B}} = \operatorname{diag}(1, -1, 1, -1),$$

and

$$[J]_{\mathcal{B}} = \operatorname{diag}(j, j),$$

where $j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

From above corollary we have the 3-form θ vanishes on $\mathfrak{g}$. In particular, we have $\theta(U, V, JV) = 0$ for all U, V in $\mathfrak{g}$; equivalently

$$\langle [V, U], V \rangle \quad = \quad -\langle [V, JU], JV \rangle.$$

It follows

$$\begin{aligned}
[X, JX] &= a\, Y + b\, JY, \\
[X, Y] &= t_1\, X + t_2\, JX + t_3\, Y + t_4\, JY, \\
[X, JY] &= -t_2\, X + t_1\, JX + t_5\, Y + t_6\, JY, \\
[JX, Y] &= t_7\, X + t_8\, JX - t_4\, Y + t_3\, JY, \\
[JX, JY] &= -t_8\, X + t_7\, JX - t_6\, Y + t_5\, JY, \\
[Y, JY] &= c\, X + d\, JX,
\end{aligned} \tag{15}$$

Besides, by using again the 3-form θ, since $\theta(U, U, V) = 0$ for all $U, V \in \mathfrak{g}$, we obtain

$$\begin{aligned}
a &= t_2 + t_7, & b &= t_8 - t_1, \\
c &= -(t_4 + t_5), & d &= t_3 - t_6.
\end{aligned} \tag{16}$$

Now, computing all the Jacobi equations involving the elements of $\mathcal{B}$ and considering the equalities given in (15) we obtain the following equations

$$\begin{array}{rcl} -(t_8+t_1)\,[X,JX]-b\,[Y,JY]+t_3\Delta(X,Y)+t_4\Delta(X,JY) & = & 0, \\ (t_2-t_7)\,[X,JX]+a\,[Y,JY]+t_5\Delta(X,Y)+t_6\Delta(X,JY) & = & 0, \\ -d\,[X,JX]+(t_3+t_6)\,[Y,JY]-t_1\Delta(X,Y)+t_2\Delta(X,JY) & = & 0, \\ -c\,[X,JX]+(t_4-t_5)\,[Y,JY]+t_7\Delta(X,Y)-t_8\Delta(X,JY) & = & 0, \end{array} \tag{17}$$

where $\Delta(U,V)=[JU,V]-[U,JV]$, for all $U,V\in\mathfrak{g}$.

If $\{[X,JX]\,,[Y,JY]\,,\Delta(X,Y),\Delta(X,JY)\}$ is a linearly independent set, then we have that all the constants are zero and so $\mathfrak{g}$ is the four dimensional abelian Lie algebra.

From here on, we assume that this set is linearly dependent and we study the set of the 4-uplas $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ satisfying

$$\lambda_1[X, JX] + \lambda_2[Y, JY] + \lambda_3\Delta(X, Y) + \lambda_4\Delta(X, JY) = 0.$$

This set coincides with the set of solutions of the homogeneous linear system $Ax = 0$, where

$$A = \begin{pmatrix} 0 & c & a & -b \\ 0 & d & b & a \\ a & 0 & c & d \\ b & 0 & d & -c \end{pmatrix}.$$

We have two cases to analize:

Case 1. If at least one of the coefficients of A is non zero, we have that the Lie algebra is

$$\mu_{a,b,\varepsilon} = \begin{cases} [X, JX] &= a\,Y + b\,JY, \\ [X, Y] &= a\,JX - \varepsilon b\,JY, \\ [X, JY] &= -a\,X + \varepsilon a\,JY, \\ [JX, Y] &= b\,JX + \varepsilon b\,Y, \\ [JX, JY] &= -b\,X - \varepsilon a\,Y, \\ [Y, JY] &= \varepsilon b\,X - \varepsilon a\,JX \end{cases} \tag{18}$$

with $\varepsilon^2 = 1$ and $a, b \in \mathbb{R}$ $(a, b \neq 0)$. To prove this assertion, we consider the solutions to $Ax = 0$ from above

$$v_1 = \begin{pmatrix} -t_1 - t_8 \\ t_1 - t_8 \\ t_3 \\ t_4 \end{pmatrix}, \; v_2 = \begin{pmatrix} t_2 - t_7 \\ t_2 + t_7 \\ t_5 \\ t_6 \end{pmatrix}, \; v_3 = \begin{pmatrix} t_6 - t_3 \\ t_3 + t_6 \\ -t_1 \\ t_2 \end{pmatrix} \; v_4 = \begin{pmatrix} t_4 + t_5 \\ t_4 - t_5 \\ t_7 \\ -t_8 \end{pmatrix}.$$

It is easy to see that $\mu_{a,b,\varepsilon}$ is isomorphic to the Lie algebra

$$\mathfrak{r}_{-1,-1} \quad = \quad \{[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_1, e_4] = -e_4.$$

Case 2. On the other hand, if all the coefficients of A are zero, we have that $\mathfrak{g}$ is the real Lie algebra underlying on the 2-dimensional complex Lie algebra $\mathfrak{aff}(\mathbb{C})$. Indeed, $a = b = c = d = 0$ implies that $t_5 = -t_4$, $t_6 = t_3$, $t_7 = -t_2$ and $t_8 = t_1$, and hence

$$\mu_{t_1,t_2,t_3,t_4} = \begin{cases} [X, JX] &= 0, \\ [X, Y] &= t_1 X + t_2 JX + t_3 Y + t_4 JY, \\ [X, JY] &= -t_2 X + t_1 JX - t_4 Y + t_3 JY, \\ [JX, Y] &= -t_2 X + t_1 JX - t_4 Y + t_3 JY, \\ [JX, JY] &= -t_1 X - t_2 JX - t_3 Y - t_4 JY, \\ [Y, JY] &= 0. \end{cases} \tag{19}$$